2022 CMWMC Individual Round Solutions

1. Let
$$x = \frac{15213}{15 - 213}.$$
Find the integer nearest to x.

Proposed by Justin Hsieh

Answer. -77

Solution. We can compute $15 - 213 = -198 = 33 \times (-6)$, and $15213 = 33 \times 461$. The desired quotient is $461/(-6) \approx -76.83$, and the nearest integer to this is $\boxed{-77}$.

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2. A grocery store sells oranges for either \$1 each or five for \$4. If Theo wants to buy 40 oranges, they would save $\$k$ by buying all five-packs instead of all single oranges. What is k?

Proposed by Connor Gordon

Answer. 8

Solution. The individual oranges would cost $40 \cdot \$1 = \$40$. If they bought five-packs instead, they would need $40/5 = 8$ such packs, for a total cost of $8 \cdot \$4 = \$32$. This corresponds to a savings of $\$40 - \$32 = \$\boxed{8}$.

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3. Let $ABCD$ be a square. If AB and CD were increased in length by 20% and AD and BC were decreased in length by 20% while keeping $ABCD$ a rectangle, the area of $ABCD$ would change by $k\%$. Find k.

Proposed by Connor Gordon

Answer. -4 (4 also accepted)

Solution. Letting the original side length be s (and thus the original area s^2), the resulting rectangle would have side lengths $\frac{6}{5}s$ and $\frac{4}{5}s$ and thus area $\frac{24}{25}s^2$, which is a change of $-\frac{1}{25}$ of the original area s^2, or $\boxed{-4}\%$.

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4. Polly writes down all nonnegative integers that contain at most one 0, at most three 2s, and no other digits. What is the median of all numbers that Polly writes down?

Proposed by Justin Hsieh

Answer. 211

Solution. The numbers written are

$$0, 2, 20, 22, 202, 220, 222, 2022, 2202, 2220$$

and the middle two are 202 and 220, so the median is $\boxed{211}$.

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5. Let P be a point. 7 circles of distinct radii all pass through P. Let n be the total number of intersection points, including P. What is the ratio of the maximum possible value of n to the minimum possible value of n?

   Proposed by Puhua Cheng

   Answer. 22

   Solution. If all circles are tangent at P, n attains its minimum at 1. To compute the maximum, note that every pair of circles intersect at at most one other point other than P. Therefore the maximum number of intersections is $\binom{7}{2} + 1 = \boxed{22}$

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6. Define the sequence $\{a_n\}$ recursively with $a_0 = 2, a_1 = 3$, and

$$a_n = a_0 + \cdots + a_{n-1},$$

   for $n \geq 2$. What is a_{2022}?

   Proposed by Max Grebinskiy

   Answer. $5 \cdot 2^{2020}$

   Solution. We claim that $a_n = 5 \cdot 2^{n-2}$ for $n \geq 2$. We prove is through induction. The base case is $n = 2$. Then $a_2 = a_1 + a_0 = 5 = 5 \cdot 2^{2-2}$, so the base case holds. Now assume that $a_i = 5 \cdot 2^{i-2}$ for $2 \leq i \leq n$, where n is fixed. We want to show that it holds true for $n + 1$. Then

$$\begin{aligned} a_{n+1} &= a_n + \cdots + a_2 + a_1 + a_0 \\ &= 5 \cdot 2^{n-2} + \cdots + 5 \cdot 2^0 + 3 + 2 \\ &= 5(2^{n-2} + \cdots + 2^0 + 1) \\ &= 5(2^{n-1}). \end{aligned}$$

   so the inductive step holds. Then the induction is complete, so $a_n = 5 \cdot 2^{n-2}$. We plug in $n = 2022$ to get $a_{2022} = \boxed{5 \cdot 2^{2020}}$

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7. Define the sequence $\{a_n\}$ recursively with $a_0 = 1$, $a_1 = 0$, and

$$a_n = 2a_{n-1} + 9a_{n-2}$$

   for all $n \geq 2$. What is the units digit of a_{2022}?

Proposed by Joy Song

Answer. 9

Solution. The units digits repeat in the sequence

$$1, 0, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 9, 8, \ldots$$

So $A[i] \equiv 1 - i \mod 10$. Plugging in $i = 2022$ gives a last digit of $\boxed{9}$.

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8. Suppose that x satisfies $|2x - 2| - 2 \leq x$. Find the sum of the minimum and maximum possible value of x.

   Proposed by Joy Song

   Answer. 4

   Solution. Let us case on the sign of $2x - 2$. Assuming $x \geq 1$, we are interested in when $2x - 4 \leq x$, which occurs when $x \leq 4$. Assuming $x \leq 1$, we are interested in when $-2x \leq x$, which occurs when $x \geq 0$. So, we have that the min and max are $0, 4$, respectively, and the answer is $\boxed{4}$.

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9. Clarabelle has 5000 cards numbered 1 to 5000. They pick five at random and then place them face down such that their numbers are increasing from left to right. They then turn over the third card to reveal the number 2022. What is the probability that the first card is a 1?

   Proposed by Connor Gordon

   Answer. $\frac{2}{2021}$

   Solution. There are $\binom{2021}{2}$ possible configurations for the first two cards. If the first card is a 1, then there are 2020 options for the second card. Since all of the possible configurations are equally likely, this gives a probability of $2020/\binom{2021}{2} = \boxed{\frac{2}{2021}}$.

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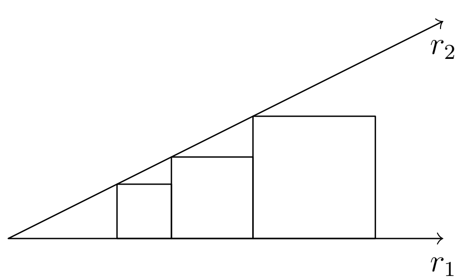
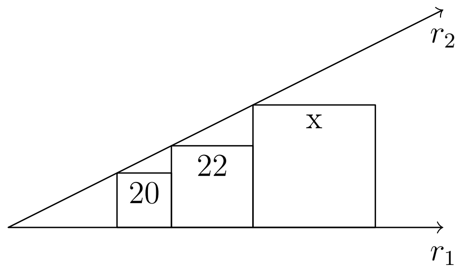
10. Rays r_1 and r_2 share a common endpoint. Three squares have sides on one of the rays and vertices on the other, as shown in the diagram. If the side lengths of the smallest two squares are 20 and 22, find the side length of the largest square.



Proposed by Connor Gordon

Answer. 121/5

Solution. Let the answer be x, so the side lengths look like this:



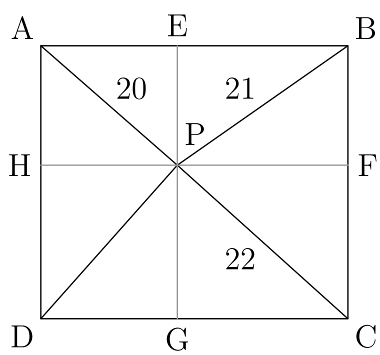
Observe that the triangles above the first two squares are similar. The first triangle has horizontal and vertical legs of length 20 and $22 - 20 = 2$, respectively. The second triangle has horizontal and vertical legs of length 22 and $x - 22$, respectively. Therefore, $x = 22 \cdot \frac{2}{20} + 22 = \frac{11}{5} + 22 = \boxed{\frac{121}{5}}$.

11. There exists a rectangle $ABCD$ and a point P inside $ABCD$ such that $AP = 20$, $BP = 21$, and $CP = 22$. In such a setup, find the square of the length DP. In other words, compute DP^2.

Proposed by Connor Gordon

Answer. 443

Solution. Let the answer be x, so the side lengths look like this:



Using the gray lines, one can see by the Pythagorean theorem that

$$AP^2 + CP^2 = EP^2 + FP^2 + GP^2 + HP^2 = BP^2 + DP^2$$

Therefore,

$$DP^2 = AP^2 + CP^2 - BP^2 = 400 + 484 - 441 = \boxed{443}.$$

(Sidenote: The above result is usually called the British Flag Theorem)

12. Compute the smallest integer N such that $5^6 = 15625$ appears as the last five digits of 5^N, where $N > 6$.

Proposed by Xuyuan Chen

Answer. 14

Solution. Since we know that 5^6 appears as the last few digits of 5^N, $5^N - 5^6$ must be divisible by 10^5. Since N is greater than 6, we can factor out 5^6 from 5^N to get $5^6 \cdot (5^{N-6} - 1)$. If this is divisible by 10^5, then 2^5 divides $5^{N-6} - 1$.

It is now straightforward to list out powers of 5 modulo 32, and one can see that $5^8 - 1$ is the first to be divisible by 32. Hence, the answer is $6 + 8 = \boxed{14}$.

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13. There exist two complex numbers z_1, z_2 such that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 338$$

Find the length of the hypotenuse of the right triangle formed with legs of length $|z_1|, |z_2|$.

Proposed by Michael Duncan

Answer. Using the identity $|z_1|^2 = z_1 \bar{z_1}$, we can simplify the entire expression to

$$2|z_1|^2 + 2|z_2|^2 = 338 \implies |z_1|^2 + |z_2|^2 = 169$$

Thus the answer is $\boxed{13}$.

(Sidenote: The above result is more generally known as the parallelogram law)

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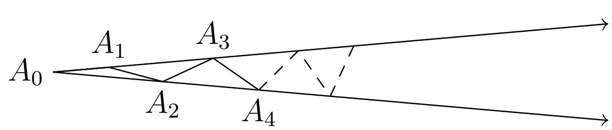
14. Blåhaj has two rays with a common endpoint A_0 that form an angle of 1°. They construct a sequence of points $A_0, \ldots, A_n$ such that for all $1 \le i \le n$, $|A_{i-1}A_i| = 1$, and $|A_iA_0| > |A_{i-1}A_0|$. Find the largest possible value of n.



Proposed by Connor Gordon

Answer. 90

Solution. Starting with the second, each new line segment encloses a new isosceles triangle. A simple angle chase yields successive base angles of 1°, 2°, 3°, and so on in each of these triangles. We can go as far as a base angle of 89°, as trying to go further will cause us to repeat points, which does not increase distance from A_0. The 89th triangle appears when the 90th segment does, meaning the maximal n is $\boxed{90}$.

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15. Consider the sequence $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots$ Find the sum of the first 100 terms of the sequence.

Proposed by Ishin Shah

Answer. 500

Solution. We first find the number of 'groups'. Note that $\binom{14}{2} = 91$, so we have 13 complete groups, each of sum $\binom{k}{2}$, for k ranging from 2 to 14. By the hockey-stick identity, we have

$$\sum_{k=2}^{14} \binom{k}{2} = \binom{15}{3} = 455.$$

Finally, our remaining terms are 1 to 9, so our total sum is $\boxed{500}$.

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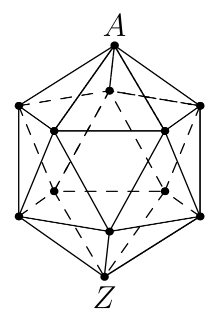
16. Suppose Annie the Ant is walking on a regular icosahedron (as shown). She starts on point A and will randomly create a path to go to point Z which is the point directly opposite to A. Every move she makes never moves further from Z, and she has equal probability to go down every valid move. What is the expected number of moves she can make?



Proposed by Ishin Shah

Answer. 6

Solution. Lets categorize the vertices into 4 levels, defined by distance to A. Due to the condition that she never moves further from Z, she can never go to a higher level, so she must stay on her level or progress to a lower level. At the first level, she can only make a move to level two, which is 1 move. At level two, she always has 4 moves she can make, 2 of them move down. At level three, she always has 3 moves she can make, 1 of which moves down.

Let E_n be the expected number of moves to move from level n to $n+1$. We have $E_n = 1$,

$$E_2 = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (E_2 + 1) \text{ and } E_3 = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (E_3 + 1).$$

This solves to $E_2 = 2, E_3 = 3$. The final expected value is, by linearity of expected value, $\boxed{6}$.

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17. Suppose that z is a complex number, where the expression

$$\frac{z-2i}{z+1}$$

is real. Find $\min|z-1|$.

Proposed by Francesca Yu

Answer. $\frac{4}{\sqrt{5}}$

Solution. $(z-2i)/(z+1)$ is real means that the argument of $z-2i$ and $z+1$ differ by a multiple of $180°$, or that $z-2i$ and $z+1$ must be colinear with the origin. Hence, z must lie on the lie joined by $-2i$ and 1. $\min|z-1|$ is the distance from 1 to this line, which is $\boxed{\frac{4}{\sqrt{5}}}$

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18. Scotty has a circular sheet of paper with radius 1. They split this paper into n congruent sectors, and with each sector, tape the two straight edges together to form a cone. Let V be the combined volume of all n cones. What is the maximum value of V?

Proposed by Lohith Tummala

Answer. $\frac{\pi}{\sqrt{3}/12}$

Solution. When we split the circle into n sectors, we see that each sector has a central angle of $360/n$ Thus, the arc length is

$$\frac{360/n}{360}\cdot 2\pi = \frac{2\pi}{n}$$

This arc length is the circumference of the circular cone base formed by the sector. This base has a radius of $1/n$. Using Pythagorean Theorem on the right triangle with hypotenuse 1 and a leg being a radius of $1/n$ we get the height to be

$$\sqrt{1^2-\left(\frac{1}{n}\right)^2} = \frac{\sqrt{n^2-1}}{n}$$

The volume of each cone is

$$\frac{\pi}{3}\left(\frac{1}{n}\right)^2\cdot\frac{\sqrt{n^2-1}}{n} = \frac{\pi\sqrt{n^2-1}}{3n^3}$$

So,

$$V = \frac{\pi\sqrt{n^2-1}}{3n^2}.$$

Notice that as n increases, the V gets lower. Thus, we take the first value of n that results in a positive V. This occurs when $n=2$, when $V = \boxed{\frac{\pi\sqrt{3}}{12}}$.

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19. Let

$$P(x) = (x-3)^m \left(x - \frac{1}{3}\right)^n$$

where m, n are positive integers. How many ordered pairs (m, n) for $m, n \le 100$ result in $P(x)$ having integer coefficients for its first three terms and last term? Assume $P(x)$ is depicted from greatest to least exponent of x.

Proposed by Kevin He

Answer. 517

Solution. For the last term to have integer coefficients, it suffices that $m \ge n$. For the first three terms, by the binomial theorem,

$$(x-3)^m = x^m - 3mx^{m-1} + 9\binom{m}{2}x^{m-2} + \dots$$

$$\left(x - \frac{1}{3}\right)^n = x^n - \frac{n}{3}x^{n-1} + \frac{1}{9}\binom{n}{2}x^{n-2} + \dots$$

So, the first three terms of $P(x)$ are

$$x^{m+n} + \left(-3m - \frac{n}{3}\right)x^{m+n-1} + \left(\frac{1}{9}\binom{n}{2} + mn + 9\binom{m}{2}\right)x^{m+n-2}$$

The coefficient of the 1st term is 1, an integer. The coefficient of the 2nd term is an integer when $3 \mid n$. For the third term, it suffices that $9 \mid \binom{n}{2}$, which occurs when $9 \mid n$ or $9 \mid n-1$. But $9 \nmid n-1$ since we must have $3 \mid n$ from the 2nd term. So, our overall conditions are $9 \mid n$ and $m \ge n$.

Now, $100 - 9 + 1 = 92$ values for m work for $n = 9$, $100 - 18 + 1 = 83$ values for m work for $n = 18$, and so on until we get that $100 - 99 + 1 = 2$ values for m work for $n = 99$. $2 + 11 + \cdots + 92 = \frac{94}{2}(11) = \boxed{517}$.

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20. Let $f(x) = |x| - 1$ and $g(x) = |x - 1|$. Define

$$f^n(x) = \underbrace{f(f(f(\ldots f}_{n \text{ times}}(x))),$$

and define $g^n(x)$ similarly. Let the number of solutions to $f^{20}(x) = 0$ and $g^{20}(x) = 0$ be a, b, respectively. Find $a - b$.

Proposed by Max Grebinskiy

Answer. 1

Solution. We can directly find a, b seperately.

Consider $g^{20}(x) = ||\cdots|x-1|-1|-\cdots|-1|$, where there are twenty -1's. Note that in general, if $x - 1 = k > 0$, then the inner expression of $g^{20}(x)$ continues to decrease until it reaches 0, at which point it cycles between 0 and 1. Similarly, if $x - 1 = j < 0$, then $|x-1| = -j > 0$, and the inner expression decreases until it reaches 0 and begins to cycle again.

We first case on if $x \geq 20, x \leq -18$, or if $-18 < x < 20$. If $x \geq 20$, then $g^{20}(x) = g^{19}(g(x)) = g^{19}(|x-1|) = g^{19}(x-1)$, where $x - 1 \geq 19$. This can be applied iteratively to see that $g^{20}(x) = x - 20 \geq 0$, so $x = 20$ is the only solution in this case. If $x \leq -18$, then $g^{20}(x) = g^{19}(g(x)) = g^{19}(|x-1|) = g^{19}(-x+1)$, where $-x+1 \geq 19$. Then $g^{19}(-x+1) \geq 0$ for all $x \leq -18$, and it can be shown that $g^{19}(x) = (-x+1) - 19 = -x - 18 \geq 0$, so $x = -18$ is the only solution in this case. Now assume that $-18 < x < 20$. It is straightforward that only integer solutions exist. It suffices to show that all even solutions work, and no odd solutions work. Note that the parity of $g^{20}(x)$ is equal to the parity of $g^{18}(x), g^{16}(x), \cdots, g^{2}(x)$. Then if x is odd, then $g^2(x)$ is also odd, which implies that $g^{20}(x)$ is odd, so $g^{20}(x) \neq 0$. If x is even, then $g^2(x)$ is even, which implies $g^{20}(x)$ is even. However, from our initial note, $g^{20}(x)$ approaches 0 and then cycles between 0 and 1. As the parity is even, then $g^{20}(x) = 0$ for all even x in $-18 < x < 20$. Overall, $n = 20$ solutions.

The analysis of $f^{20}(x)$ is very similar, except now $f^{20}(x)$ continues to decrease until it reaches -1, and then it cycles between 0 and -1. We case on if $x \geq 20, x \leq -20$, or if $-20 < x < 20$. If $x \geq 20$, then $f^{20}(x) = x - 1 - 1 \cdots - 1 = x - 20 \geq 0$, so $x = 20$ is the only solution. If $x \leq -20$, then $f(x) = |x| - 1 = -x - 1 \geq 19$, and so $f^{20}(x) = -x - 1 - 1 \cdots - 1 = -x - 20 \geq 0$, so $x = -20$ is the only solution. If $-20 < x < 20$, then the parity of $f^{20}(x)$ has the same parity as $f^{18}(x), \cdots, f^{2}(x)$, so it can be seen that if x is odd, then $f^2(x)$ has odd parity again, and $f^{20}(x)$ has no solutions. If x is even, then $f^2(x)$ has even parity, and $f^{20}(x)$ also has even parity. Then as $f^{20}(x)$ is bounded between -1 and 0 for $-20 < x < 20$, for all even x, we must have $f^{20}(x) = 0$. Then there are a total of $m = 21$ solutions, so $m - n = \boxed{1}$.

In general, for any $f^k(x), g^k(x)$, we have that $m - n = 1$.